Research Journal of Recent Sciences ISSN 2277-2502
Vol. **2(4),** 91-93, April **(2013)** *Res. J. Recent Sci.*

Some Properties of AG*-groupoid

Ahmad I.[1], Rashad M.[1] and Shah M.[2]
[1]Department of Mathematics, University of Malakand, Khyber Pakhtunkhwa, PAKISTAN
[2]Department of Mathematics, Abdul Wali Khan University Mardan, Khyber Pakhtunkhwa, PAKISTAN

Available online at: www.isca.in
Received 1st November 2012, revised 2nd December 2012, accepted 5th January 2013

Abstract

We prove some new results for AG-groupoid such as: (1) an AG*-groupoid is a Bol*-AG-groupoid, (2) an AG*-groupoid is nuclear square AG-groupoid, (3) a cancellative AG*-groupoid is transitively commutative AG-groupoid, (4) a T^1-AG-3-band is AG*-groupoid, (5) an AG-groupoid with a left cancellative element is a -AG-groupoid, and (6) an AG*-groupoid is left alternative AG-groupoid.*

Keywords: AG-groupoid, AG*-groupoid, AG-group, types of AG-groupoid, paramedial AG-groupoid, Nuclear square AG-groupoid.

Introduction

A groupoid $(S,.)$ or shortly S is called an Abel Grassmann groupoid or shortly AG-groupoid (also called a left almost semigroup[1], shortly LA-semigroup) if it satisfies the left invertive law: $(ab)c = (cb)a$. Other names such as left invertive groupoid[2], and right modular groupoid[3] can also be found for this structure in the literature. Clearly every commutative semigroup satisfies the left invertive law and thus becomes an AG-groupoid and so an AG-groupoid generalizes commutative semigroup. An AG-groupoid satisfying the identity $(ab)c = b(ac)$ is AG*-groupoid. Recently some new classes[4-7] of AG-groupoid have been defined. In this paper we investigate some interesting relations of AG*-groupoid with some of those new classes. We first recall some necessary definitions before to discuss in detail the results that we have listed in the abstract.

An AG-groupoid S is called:
−an AG-3-band if $a(aa) = (aa)a = a \; \forall \, a \in S$.
−a transitively commutative AG-groupoid if $\forall \, a,b,c \in S, ab = ba, bc = cb \Rightarrow ac = ca$.
−a T^1-AG-groupoid if $\forall \, a,b,c,d \in S, ab = cd \Rightarrow ba = dc$.
−a T^3-AG-groupoid if $\forall \, a,b,c \in S, ab = ac \Rightarrow ba = ca$.
−a Bol*-AG-groupoid if it satisfies the identity $a(bc.d) = (ab.c)d$.
−a paramedial-AG-groupoid if $\forall \, a,b,c,d \in S, (ab)(cd) = (db)(ca)$.
−a left nuclear square AG-groupoid[8] if $\forall \, a,b,c \in S \; a^2(bc) = (a^2b)c$.
−a right nuclear square AG-groupoid[8] if $\forall \, a,b,c \in S \; (ab)c^2 = a(bc^2)$.
−a middle nuclear square AG-groupoid[8] if $\forall \, a,b,c \in S \; (ab^2)c = a(b^2c)$.
−a nuclear square AG-groupoid if it is left, right and middle nuclear square AG-groupoid[8].
−a left alternative AG-groupoid if $\forall \, a,b \in S, aa.b = a.ab$.

Let S be an AG-groupoid. An element a of S is called left cancellative if $ax = ay \Rightarrow x = y \;\; \forall \, x,y \in S$. Similarly an element a of S is called right cancellative if $xa = ya \Rightarrow x = y \;\; \forall x,y \in S$. An element a of S is called cancellative if it is both left and right cancellative. S is called left cancellative (right cancellative, cancellative) if every element of S is left cancellative (right cancellative, cancellative). For detailed study of cancellativity of AG-groupoids we refer the reader to reference[9]. It has been proved that a Bol*-AG-groupoid is paramedial AG-groupoid and a paramedial AG-groupoid is left nuclear square[6]. Recall that an AG-groupoid S always satisfies the medial law[1]: $(ab)(cd) = (ac)(bd)$. It can be easily verified that a T^1-AG-groupoid is a T^3-AG-groupoid.

Some Properties of AG*-groupoid
We start with the following theorem that gives a relation between AG* -groupoids and Bol*-AG-groupoids.

Theorem 1: *Every AG*-groupoid is Bol* -AG-groupoid.*
Proof: Let S be an AG* -groupoid, and let $a,b,c,d \in S$. Then by definition of AG*-groupoid $(ab)c = b(ac)$. Now
$(ab.c)d = dc.ab$ (by left invertive law)
$\Rightarrow \; (ab.c)d = da.cb$ (by medial law)
$\Rightarrow \; (ab.c)d = a(d.cb)$ (by definition of AG*-groupoid)
$\Rightarrow \; (ab.c)d = a(cd.b)$ (by definition of AG*-groupoid)
$\Rightarrow \; (ab.c)d = a(bd.c)$ (by left invertive law)
$\Rightarrow \; (ab.c)d = a(d.bc)$ (by definition of AG*-groupoid)
$\Rightarrow \; (ab.c)d = da.bc$ (by definition of AG*-groupoid)
$\Rightarrow \; (ab.c)d = (bc.a)d$ (by left invertive law)
$\Rightarrow \; (ab.c)d = a(bc.d)$ (by definition of AG*-groupoid)
Hence AG* -groupoid is Bol* -AG-groupoid.

Here is a counterexample to show that the converse of the previous theorem is not valid i.e. every Bol* -AG-groupoid is not AG* -groupoid.

Example 1: Bol* -AG-groupoid of order 3 that is not AG* groupoid.

*	1	2	3
1	1	1	1
2	1	1	1
3	1	2	1

Clearly $(3 * 3) * 2 \neq 3 * (3 * 2)$.

Corollary 1: *Every AG* -groupoid is paramedial AG-groupoid.*

Corollary 2: *Every AG* -groupoid is left nuclear square AG-groupoid.*

The following theorem proves that in fact an AG*-groupoid is nuclear square AG-groupoid.

Theorem 2: Let S be an AG* -groupoid. Then i. S is middle nuclear square AG-groupoid, ii. S is right nuclear square AG-groupoid, iii. S is nuclear square AG-groupoid.

Proof. (i). Let $a, b, c \in S$. Then
$(ab^2)c = b^2(ac)$ (by definition of AG*-groupoid)
$\Rightarrow (ab^2)c = bb.ac$
$\Rightarrow (ab^2)c = ba.bc$ (by medial law)
$\Rightarrow (ab^2)c = a(b.bc)$ (by definition of AG*-groupoid)
$\Rightarrow (ab^2)c = a(bb.c)$ (by definition of AG*-groupoid)
$\Rightarrow (ab^2)c = a(b^2c)$.
Hence S is middle nuclear square AG-groupoid.
(ii). Let $a, b, c \in S$. Then
$a(bc^2) = ba.c^2$ (by definition of AG*-groupoid)
$\Rightarrow a(bc^2) = bc.ac$ (by medial law)
$\Rightarrow a(bc^2) = (ac.c)b$ (by left invertive law)
$\Rightarrow a(bc^2) = (cc.a)b$ (by left invertive law)
$\Rightarrow a(bc^2) = a(cc.b)$ (by definition of AG*-groupoid)
$\Rightarrow a(bc^2) = a(c.cb)$ (by definition of AG*-groupoid)
$\Rightarrow a(bc^2) = ca.cb$ (by definition of AG*-groupoid)
$\Rightarrow a(bc^2) = c^2.ab$ (by medial law)
$\Rightarrow a(bc^2) = (ab.c)c$ (by left invertive law)
$\Rightarrow a(bc^2) = (b.ac)c$ (by definition of AG*-groupoid)
$\Rightarrow a(bc^2) = ac.bc$ (by definition of AG*-groupoid)
$\Rightarrow a(bc^2) = (ab)c^2$. (by medial law)
Hence S is right nuclear square AG-groupoid.
(iii). From (i), (ii) and Corollary 2, the result follows.

The following theorem shows that if we allow cancellativity to an AG*-groupoid, it becomes a transitively commutative AG-groupoid.

Theorem 4: *Every cancellative AG* -groupoid S is transitively commutative AG-groupoid.*

Proof: Let S be an AG* -groupoid, and let $a, b, c \in S$, such that $ab = ba$, $bc = cb$. Then
$(ac)b = c(ab)$ (by definition of AG*-groupoid)
$\Rightarrow (ac)b = c(ba)$ (as $ab = ba$)
$\Rightarrow (ac)b = (bc)a$ (by definition of AG*-groupoid)
$\Rightarrow (ac)b = (cb)a$ (as $bc = cb$)
$\Rightarrow (ac)b = (ab)c$ (by left invertive law)
$\Rightarrow (ac)b = (ba)c$ (as $ab = ba$)
$\Rightarrow (ac)b = (ca)b$ (by left invertive law)
$\Rightarrow ac = ca.$ (by right cancellativity)

Hence cancellative AG* -groupoid is transitively commutative-AG-groupoid.

Next we have the following:

Theorem 5: *An AG* -groupoid S having a left cancellative element is T^1 - AG-groupoid.*

Proof: Let x be a left cancellative an AG* -groupoid S and let $a, b, c, d \in S$. Let $ab = cd$. Then
$x(ba) = (bx)a$ (by definition of AG*-groupoid)
$\Rightarrow x(ba) = (ax)b$ (by left invertive law)
$\Rightarrow x(ba) = x(ab)$ (by definition of AG*-groupoid)
$\Rightarrow x(ba) = x(cd)$ (as $ab = cd$)
$\Rightarrow x(ba) = (cx)d$ (by definition of AG*-groupoid)
$\Rightarrow x(ba) = (dx)c$ (by left invertive law)
$\Rightarrow x(ba) = x(dc)$ (by definition of AG*-groupoid)
$\Rightarrow ba = dc.$ (by left cancellativity)

Hence left cancellative AG* -groupoid is T^1 -AG-groupoid.

Corollary 3: *An AG* -groupoid S having a cancellative element is T^3 - AG-groupoid.*

Here is a counterexample showing that a Bol*-AG-groupoid is not necessarily *T^3- AG-groupoid.*

Example 2: *An AG* -groupoid of order 4 which is neither T^3-AG-groupoid nor T^1-AG-groupoid.*

*	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	2	1
4	1	1	2	2

The above example shows that every AG* -groupoid is not a T^1-AG-groupoid, however we have the following:

Theorem 7: *Every T^1 -AG-3-band is AG* -groupoid.*

Proof: Let S be a T^1 -AG-groupoid which is also AG-3-band and let $a, b, c, d \in S$. Then by definition of T^1 -AG-groupoid, $ab = cd \Rightarrow ba = dc$.

Now since,

$(ab)c = (cb)a$ (by left invertive law)

$\Rightarrow (ab)c = (cb)\big((aa)a\big)$ (by definition of AG-3-band)

$\Rightarrow (ab)c = c(aa).ba$ (by medial law)

$\Rightarrow c(ab) = ba.c(aa)$ (by definition of T^1-AG-groupoid)

$\Rightarrow c(ab) = \Big(\big(c(aa)\big)a\Big)b$ (by left invertive law)

$\Rightarrow (ab)c = b\Big(\big(c(aa)\big)a\Big)$ (by definition of T^1-AG-groupoid)

$\Rightarrow (ab)c = b((a(aa)c)$ (by left invertive law)

$\Rightarrow (ab)c = b(ac)$ (by definition of AG-3-band)

Thus S is AG* -groupoid.

The following theorem shows that the class of AG* -groupoids is a subclass of the left alternative AG-groupoids.

Theorem 6: *Every AG* -groupoid is left alternative AG-groupoid.*

Proof: Let S be an AG* -groupoid, and let $a, b, c \in S$. Then by definition of AG* -groupoid, we have,

$(ab)c = b(ac)$ (1)

Now replacing b by a in Equation (1), we have

$(aa)b = a(ab)$

Hence S is left alternative AG-groupoid.

Conclusion

Recently many new classes of AG-groupoids have been discovered. In this article we have investigated some relations of AG^*-groupoid to some of those newly discovered classes. Here is a quick review of our results. Every AG^*-groupoid is Bol^*-AG-groupoid and hence a paramedial AG-groupoid. Every AG^*-groupoid is nuclear square AG-groupoid. Every cancellative AG^*-groupoid is transitively commutative AG-groupoid. A T^1-AG-3-band is an AG^*-groupoid and an AG^*-groupoid becomes a T^3-AG-groupoid if it is having a left cancellative element. Finally we have shown that every AG^*-groupoid is left alternative AG-groupoid. It should be noted that the class of AG^*-groupoids can have many other relations with these classes and with other classes that we haven't considered here. Our work is only an initiative and should provide some motivation for future work.

References

1. Kazim M.A. and Naseerudin M., On almost semigroup, *Portugaliae Mathematica*. **36(1) (1977**)
2. Holgate P., Groupoids satisfying a simple invertive law, *Math. Stud*., **61**, 101-106 **(1992)**
3. Cho J.R., Pusan J. Jezek and T. Kepka, Praha, Paramedial Groupoids, *Czechoslovak Mathematical Journal*, **49(124) (1996)**
4. Shah M., Ahmad I., Ali A., On Introduction of New Classes of AG-groupoids, *Res. J. Recent Sci.,* **2(1),** 67-70 **(2013)**
5. Shah M., Ahmad I., Ali A., Discovery of New Classes of AG-groupoids, *Res. J. Recent Sci.,* **1(11),** 47-49 **(2012)**
6. Shah M., A theoretical and computational investigation of AG-groups, PhD Thesis, Department of Mathematics, Quaid-e-Azam University, Islamabad, **(2012)**
7. Ahmad I., Rashad M., and Shah M., Some new results on T^1, T^2 and T^4 AG-groupoids, *Res. J. Recent Sci.,* **2(3),** 64-66 **(2013)**
8. Rashad M., Ahmad I., and Shah M., On relations between right alternative and nuclear square AG-groupoids, *Int. Math. Forum*, **8(5)**, 237–243 **(2013)**
9. Shah M., Shah T., Ali A., On the Cancellativity of AG-Groupoids, *Int. Math. Forum*, **6(44)**, 2187 – 2194 **(2011)**