Research Journal of Mathematical and Statistical Sciences ______________________________ISSN 2320-6047
Vol. **12(1),** 1-4, January **(2024)** *Res. J. Mathematical and Statistical Sci.*

On some Properties of k-Beta Function associated with several Variables

Radhe Shyam Prajapat
Dept. of Mathematics M. L.V. Govt. PG College, Bhilwara, Rajasthan, India
rprajapat71@yahoo.in

Available online at: www.isca.in, www.isca.me
Received 19th May 2023, revised 19th September 2023, accepted 23rd November 2023

Abstract

Aim of this paper is to we will obtain the new interesting results on basis of the present research work with the help k-Gamma and k-Beta functions. We will derive the results related to k-Beta function associated with several variables including k-Gamma function. Applying multiple integral, we will obtain the results like Dirichlet's theorem for multiple integral.

Keywords: Gamma function, Beta function, multiple integral, Dritchlet's theorem.

Introduction

In 2007, R. Diaz at al. have introduced the following Pochhammer k-symbol and k-Gamma function[1,2].

$$(a)_{n,k} = \begin{cases} a(a+k)(a+2k).......(a+(n-1)k) \\ 1 \end{cases} ; \quad \begin{matrix} n \geq 1, k > 0 \\ k = 1 \end{matrix} \tag{1}$$

For $k > 0, z \in C$ and $\text{Re}(z) > 0$, the k-Gamma function is defined as

$$\Gamma_k(z) = \int_0^\infty x^{z-1} e^{-\frac{x^k}{k}} dx \tag{2}$$

Equation (2) we can write as

$$\Gamma_k(z) = k^{\frac{x}{k}} \Gamma\left(\frac{z}{k}\right) \tag{3}$$

Many researchers have discovered some properties of k-Gamma function[3-8]

$$\Gamma_k(z+k) = z\Gamma_k(z) \tag{4}$$

$$\Gamma_k(k) = 1 \quad ; \quad k>0 \tag{5}$$

$$(a)_{n,k} = \frac{\Gamma_k(z+nk)}{\Gamma_k(z)} \tag{6}$$

$$\frac{\Gamma_k(z)}{a^{\frac{z}{k}}} = \int_0^\infty x^{z-1} e^{-a\frac{x^k}{k}} dx ; \; a \in R \tag{7}$$

$$\Gamma_k(ak) = k^{a-1}\Gamma(a); \; a \in R \tag{8}$$

$$\Gamma_k(z)\Gamma_k(k-z) = \frac{\pi}{k \sin\left(\frac{\pi z}{k}\right)} \tag{9}$$

$$\Gamma_k(2z) = \sqrt{\frac{k}{\pi}}\, 2^{2\frac{x}{k}-1} \Gamma_k(z)\Gamma_k\left(z + \frac{k}{2}\right) \tag{10}$$

The k-Beta function $B_k(m,n)$ for two variable m and n is defined by[1,3,4]

$$B_k(m,n) = \frac{1}{k}\int_0^1 x^{\frac{m}{k}-1}(1-x)^{\frac{m}{k}-1} dx ; \text{Re}(m) > 0, \; \text{Re}(n) > 0 . \tag{11}$$

Equation (11) in term of Gamma function is given by

$$B_k(m,n) = \frac{\Gamma_k(m)\Gamma_k(n)}{\Gamma_k(m+n)} ; \; \text{Re}(m) > 0, \text{Re}(n) > 0 \tag{12}$$

Main results of k-Beta function associated with several variables

Let $l=(l_1,l_2,l_3,...l_n)\in C^n, l_i \neq \{0,-1,-2,...\}, n\geq 2$ then Beta function associated with n variables is defined by

$$B(l)=B(l_1,l_2,l_3,...l_n)=\frac{\Gamma l_1 \Gamma l_2 \Gamma l_3 ... \Gamma l_n}{\Gamma(l_1+l_2+l_3...+l_n)} \quad (13)$$

If put $n=2$, then (13) will be classical Beta function[9-11]

Let $l=(l_1,l_2,l_3,...l_n)\in C^n, l_i \neq \{0,-1,-2,...\}, n\geq 2$ and $k>0$ then equation (11) associated with n variable is defined by Rehman et al.[3]

$$B_k(l)=B_k(l_1,l_2,l_3,...l_n)=\frac{\Gamma_k(l_1)\Gamma_k(l_2)\Gamma_k(l_3)....\Gamma_k(l_n)}{\Gamma_k(l_1+l_2+l_3....+l_n)} \quad (14)$$

Theorem-1: Let $l=(l_1,l_2,l_3...l_n)\in C^n, l_i \neq \{0,-1,-2,.....\}$, $n\geq 2$ and $k>0$ the equation (11) has properties

$$B_k(l_1+l_2,l_2+l_3,l_3+l_1)=\frac{B_k(l_1,l_2,l_3)}{B_k(l_1,l_2)B_k(l_2,l_3)B_k(l_3,l_1)}\sqrt{\frac{\pi}{k}}\,2^{-\frac{2}{k}(l_1+l_2+l_3)+1}\frac{\Gamma_k(l_1)\Gamma_k(l_2)\Gamma_k(l_3)}{\Gamma_k\left(l_1+l_2+l_3+\frac{k}{2}\right)} \quad (15)$$

$$B_k(l_1+l_2,l_2+l_3,l_3+l_4,.....,l_n+l_1)=$$

$$\frac{B_k(l_1,l_2,l_3,....l_n)}{B_k(l_1,l_2)B_k(l_2,l_3)B_k(l_3,l_4).....B(l_n,l_1)}\sqrt{\frac{\pi}{k}}\,2^{-\frac{2}{k}(l_1+l_2+l_3....+l_n)+1}\frac{\prod_{i=1}^{n}\Gamma_k(l_i)}{\Gamma\left(l_1+l_2+l_3...+l_n+\frac{k}{2}\right)} \quad (16)$$

Proof

(i) Using equation (14) in L.H.S of equation (15)

$$B_k(l_1+l_2,l_2+l_3,l_3+l_1)=\frac{\Gamma_k(l_1+l_2)\Gamma_k(l_2+l_3)\Gamma_k(l_3+l_1)}{\Gamma_k(2[l_1+l_2+l_3])}$$

$$=\frac{(\Gamma_k(l_1))^2(\Gamma_k(l_2))^2(\Gamma_k(l_3))^2}{B_k(l_1,l_2)B_k(l_2,l_3)B_k(l_3,l_1)\sqrt{\frac{k}{\pi}}\,2^{2\frac{l_1+l_2+l_3}{k}-1}\Gamma_k(l_1+l_2+l_3)\Gamma_k\left(l_1+l_2+l_3+\frac{k}{2}\right)}$$

$$=\frac{B_k(l_1,l_2,l_3)}{B_k(l_1,l_2)B_k(l_2,l_3)B_k(l_3,l_1)}\sqrt{\frac{\pi}{k}}\,2^{-2\frac{l_1+l_2+l_3}{k}+1}\frac{\Gamma_k(l_1)\Gamma_k(l_2)\Gamma_k(l_3)}{\Gamma_k\left(l_1+l_2+l_3+\frac{k}{2}\right)}$$

Which is completes the proof of equation (15)

$$B_k(l_1+l_2,l_2+l_3,l_3+l_4,...,l_n+l_1)=\frac{\Gamma_k(l_1+l_2)\Gamma_k(l_2+l_3)\Gamma_k(l_3+l_4)....\Gamma(l_n+l_1)}{\Gamma_k 2(l_1+l_2+l_3....+l_n)}$$

$$=\frac{(\Gamma_k l_1)^2(\Gamma_k l_2)^2(\Gamma_k l_3)^2.....(\Gamma_k l_n)^2}{B_k(l_1,l_2)B_k(l_2,l_3)B_k(l_3,l_4).....B_k(l_n,l_1)\sqrt{\frac{k}{\pi}}\,2^{2\frac{\sum_{i=1}^{n}l_i}{k}-1}\Gamma_k\left(\sum_{i=1}^{n}l_i\right)\Gamma_k\left(\sum_{i=1}^{n}l_i+\frac{k}{2}\right)}$$

$$=\frac{B_k(l_1,l_2,l_3,....,l_3)}{B_k(l_1,l_2)B_k(l_2,l_3)B_k(l_3,l_4)....B_k(l_n,l_1)}2^{-2\frac{l_1+l_2+l_3....l_n}{k}+1}\frac{\prod_{i=1}^{n}\Gamma_k l_i}{\Gamma_k\left(\sum_{i=1}^{n}l_i+\frac{k}{2}\right)}$$

Which is required prove of equation (16)

Theorem-2: For $l=(l_1,l_2,l_3...l_n)\in C^n, l_i \neq \{0,-1,-2,..\}$, $n\geq 2$ and $k>0$ then equation (1.11) associated with n variables has properties

$$B_k(k+l_1,k-l_1,k+l_2,k+l_3,k-l_3,......l_n+k,k-l_n)=\frac{\prod_{i=1}^{n}l_i}{k^{2n-1}\Gamma(2n)}\left[\frac{\pi}{k}\right]^n\prod_{i=1}^{n}\cos ec\left(\frac{\pi}{k}l_i\right) \quad (17)$$

$$B_k(rk+l_1,rk-l_1,rk+l_2,rk-l_2,rk+l_3,rk-l_3,.......rk+l_n,rk-l_n)$$

$$=\frac{\prod_{i=1}^{n}(l_i)_{r,k}\left[\frac{\pi}{rk}\right]^n\prod_{i=1}^{n}\cos ec\left(\frac{\pi l_i}{rk}\right)}{k^{2nr-1}\Gamma(2nr)} \quad (18)$$

Proof

$$B_k(k+l_1,k-l_1,k+l_2,k-l_2,k+l_3,k-l_3,......l_n+k,k-l_n)$$

$$=\frac{\Gamma_k(k+l_1)\Gamma_k(k-l_1)\Gamma_k(k+l_2)\Gamma_k(k-l_2)........\Gamma_k(k+l_n)\Gamma_k(k-l_n)}{\Gamma_k(2nk)}$$

$$=\frac{l_1l_2l_3l_4...l_n\Gamma_k(l_1)\Gamma_k(k-l_1)\Gamma_k(l_2)\Gamma_k(k-l_2).....\Gamma_k(l_n)\Gamma_k(k-l_n)}{k^{2n-1}\Gamma(2n)}$$

$$=\frac{l_1l_2l_3l_4...l_n}{k^{2n-1}\Gamma(2n)}\frac{\pi}{k\sin\left(\frac{\pi l_1}{k}\right)}\frac{\pi}{k\sin\left(\frac{\pi l_2}{k}\right)}\frac{\pi}{k\sin\left(\frac{\pi l_3}{k}\right)}......\frac{\pi}{k\sin\left(\frac{\pi l_n}{k}\right)}$$

$$=\frac{\prod_{i=1}^{n}l_i}{k^{2n-1}\Gamma(2n)}\left[\frac{\pi}{k}\right]^n\prod_{i=1}^{n}\cos ec\left(\frac{\pi}{k}l_i\right)$$

Hence equation (17) has been proved

$$B_k(kr+l_1,kr-l_1,kr+l_2,kr+l_3,kr-l_3,......l_n+rk,kr-l_n)$$

$$=\frac{\Gamma_k(kr+l_1)\Gamma_k(kr-l_1)\Gamma_k(kr+l_2)\Gamma_k(kr-l_2)........\Gamma_k(kr+l_n)\Gamma_k(kr-l_n)}{\Gamma_k(2rnk)}$$

$$= \frac{(l_1)_{r,k}(l_2)_{r,k}(l_3)_{r,k}......(l_n)_{r,k}\Gamma_k(l_1)\Gamma_k(rk-l_1)\Gamma_k(l_2)\Gamma_k(rk-l_2).....\Gamma_k(l_n)\Gamma_k(rk-l_n)}{k^{2rn-1}\Gamma(2rn)}$$

$$= \frac{(l_1)_{r,k}(l_2)_{r,k}(l_3)_{r,k}.....(l_n)_{r,k}}{k^{2rn-1}\Gamma(2rn)} \frac{\pi}{rk\sin\left(\frac{\pi l_1}{rk}\right)} \frac{\pi}{rk\sin\left(\frac{\pi l_2}{rk}\right)} \frac{\pi}{rk\sin\left(\frac{\pi l_3}{rk}\right)}......\frac{\pi}{rk\sin\left(\frac{\pi l_n}{rk}\right)}$$

$$= \frac{\prod_{i=1}^{n}(l_i)_{r,k}\left[\frac{\pi}{rk}\right]^n \prod_{i=1}^{n}\cos ec\left(\frac{\pi l_i}{rk}\right)}{k^{2nr-1}\Gamma(2nr)}$$

Hence equation (18) has been proved

Dirichlet's theorem

Theorem-3: Let $\mathrm{Re}(l_1) > 0, \mathrm{Re}(l_2) > 0, \mathrm{Re}(l_3) > 0$ and $k > 0$ then equation (11) has property

$$\iiint_V x^{\frac{l_1}{k}-1} y^{\frac{l_2}{k}-1} z^{\frac{l_3}{k}-1} dxdydz = k^{\frac{l_1+l_2+l_3}{3}+3} \frac{\Gamma_k(l_1)\Gamma_k(l_2)\Gamma_k(l_3)}{\Gamma_k(l_1+l_2+l_3+k)} \quad (19)$$

Where V denotes the closed region by the bounded plane $x = 0, y = 0, z = 0, x + y + x \le k$

Proof

$$I = \iiint_V x^{\frac{l_1}{k}-1} y^{\frac{l_2}{k}-1} z^{\frac{l_3}{k}-1} dxdydz$$

$$= \int_0^k \int_0^{k-x} \int_0^{k-x-y} x^{\frac{l_1}{k}-1} y^{\frac{l_2}{k}-1} z^{\frac{l_3}{k}-1} dxdydz$$

$$= \frac{k}{l_3}\int_0^k x^{\frac{l_1}{k}-1}(k-x)^{\frac{l_2+l_3}{k}} kB_k(l_2, l_3+k)dx$$

$$= \frac{k^2}{l_3}B_k(l_2, l_3+k)k^{\frac{l_1+l_2+l_3}{k}}\int_0^1 u^{\frac{l_1}{k}-1}(1-u)^{\frac{l_2+l_3}{k}}du$$

$$= \frac{1}{l_3}k^{\frac{l_1+l_2+l_3}{k}+3}B_k(l_1, l_2+l_3+k)B_k(l_2, l_3+k)$$

$$= k^{\frac{l_1+l_2+l_3}{k}+3}\frac{\Gamma_k(l_1)\Gamma_k(l_2)\Gamma_k(l_3)}{\Gamma_k(l_1+l_2+l_k+k)}$$

Which is required the result

Thus generalized form of this theorem for more than three variables will be given as

$$\iiint_V .....\int x_1^{\frac{l_1}{k}-1} x_2^{\frac{l_2}{k}-1} x_3^{\frac{l_3}{k}-1} ....x_n^{\frac{l_n}{k}-1} dx_1dx_2...dx_n = k^{\frac{l_1+l_2+l_3.....l_n}{k}+n} \frac{\Gamma_k(l_1)\Gamma_k(l_2)\Gamma_k(l_3)...\Gamma_k(l_n)}{\Gamma_k(l_1+l_2+l_3+......l_n+k)} \quad (20)$$

Where the region of integration V will be given by $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0,......x_n \ge 0$ And $x_1 + x_2 + x_3 + ... + x_n \le k$

Corollary: For $\mathrm{Re}(m) > 0, \mathrm{Re}(n) > 0$ and $k < 0$ then

$$\iint_S x^{\frac{l_1}{k}-1} y^{\frac{l_2}{k}-1} dxdy = \frac{\alpha^{\frac{l_1}{k}}\beta^{\frac{l_2}{k}}}{2l_2}k^2 B_k\left(\frac{l_1}{2}, \frac{l_2}{2}+k\right) \quad (21)$$

Where S is taken over the positive quadrant of ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$

Conclusion

We will derive the results related to k -Beta function associated with several variables including k -Gamma function. Applying multiple integral, we will obtain the results like Dirichlet's theorem for multiple integral.

References

1. Díaz, R., & Teruel, C. (2005). q, k-Generalized gamma and beta functions. *Journal of Nonlinear Mathematical Physics*, 12(1), 118-134.
2. Diaz, R., & Pariguan, E. (2004). On hypergeometric functions and Pochhammer $ k $-symbol. arXiv preprint math/0405596.
3. Rehman, A., Mubeen, S., Safdar, R., & Sadiq, N. (2015). Properties of k-beta function with several variables. *Open Mathematics*, 13(1), 000010151520150030.
4. Kokologiannaki, C. G. & Krasniqi, V. (2013). Some properties of the k-gamma function. *Le Matematiche*, 68(1), 13-22.
5. Kokologiannaki, C. G. (2010). Properties and inequalities of generalized k-gamma, beta and zeta functions. *Int. J. Contemp. Math. Sci*, 5(14), 653-660.
6. Bapna, I. B., & Prajapat, R. S. (2020). Some Applications of Extension k-Gamma, k-Beta Functions and k-Beta Distribution. *Journal of Ramanujan Society of Mathematics and Mathematical Sciences*, 7(02), 83-98.

7. Mubeen, S., Rehman, A., & Shaheen, F. (2014). Properties of k-gamma, k-beta and k-psi functions. *Bothalia Journal*, 4, 371-379.
8. W. S. (2016). Some properties of k-gamma and k-beta functions. *In ITM Web of Conferences*, Vol. 7, 07003. EDP Sciences.
9. Prajapat, R. S. & Bapna, I. B. (2023). Some Properties of k-RIEMANN-Liouville Fractional Integral Operator. *Journal of Ramanujan Society of Mathematics & Mathematical Sciences*, 10(2).
10. Mathai, A. M., & Haubold, H. J. (2008). Special functions for applied scientists. Vol. 4. New York: Springer.
11. Carlson B. C. (1997). Special function of applied mathematics. Academic Press, New York.